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Teaching Mathematics and Computer Science

How do secondary school students from the Kurdistan Region of Iraq understand the concept of function?

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Abstract. The study investigates secondary school students' understanding of the concept of function. The paper focuses on three main aspects: students' ability to define the concept of function; students' ability to recognize different representations of function; and students' ability to convert between different representations. A test was developed to assess the three main constructs of the study and administered to 342 students in secondary schools in the Kurdistan Region of Iraq. According to the results, students have difficulties in recognizing different representations of function and conversion between them. Connections between different parts of the test may provide hints on educational challenges of how to appropriately teach functions.

Key words and phrases: secondary schools, functions, representations of functions.

MSC Subject Classification: 26Bxx, 97D60.

Introduction

A fundamental concept in Mathematics is the concept of function (Szanyi, 2019). This is a concept, as ancient as the subject of Mathematics, whereby people make a relation between two quantities (Eisenberg, 1992). In the last three decades, the mathematics education research community has focused on the concept of function as one of the basic concepts in the learning of Mathematics (Dubinsky & Harel, 1992; Evangelidou et al., 2004). Students' understanding of the concept of function is fundamental, especially for students at the secondary school level, because this concept could be used in many different mathematical subjects such as calculus. For example, in Derivative subject: $f(x) = 2x^2 + x - 1$, find dy/dx; in Integral subject: find $\int_2^4 x^3\,dx$; in Limit subject: find $\lim_{x\to 2} f(-2x^5 + 3x^2 - 7x + 5)$. To solve these questions, students have to know the concept of function, because all these examples provided in different mathematics subjects are basically functions or are explaining a part of a function. This means that if students do not understand the concept of function accurately, they face difficulties in other mathematical subjects. However, concerns have been voiced about students' difficulties in understanding this concept and recognizing different representations of it (Dubinsky & Harel, 1992). For example, Markovits et al. (1986) show that students face difficulties and confusion in representing the constant function and functions that are represented by a set of ordered points. According to Schoenfeld et al. (2019), students have difficulties in coordinating between different representations of a function such as a graph, table, or equation. Also, students cannot differentiate between the concept image and the concept definition of a function. For example, in Tall and Vinner's study (1981), students depended on the concept image to define the continuous function.

Conversely, researchers have pointed to the impact of different representations of function on students' understanding of the concept of function (Hitt, 1998; Markovits et al., 1986). Evidence from studies that have investigated this impact suggests that the use of multiple representations helps students to construct mathematical concepts (Ainsworth et al., 2002; Cramer & Henry, 2002; Jiang & McClintock, 2000; Pape & Tchoshanov, 2001). In addition, Duval's (1999) theoretical perspective illustrates that there are many types of representations of function, each one of them representing a different aspect of the concept of function. Still, none of them can describe the concept of function completely. So, students need to be familiar with all the representations of function to understand the idea of function. Greeno and Hall (1997) point out that a useful tool in building mathematical information and conceptual understanding for students is the use of different representations. The research reported above shows the necessity of students recognizing different representations of function and conversion between them in order to understand more fully the concept of function (Kurt & Cakiroglu, 2009).

Although there are many approaches to representing a function, for example, by tables, ordered pairs, verbal descriptions, and graphs, this paper focuses on three: graphs, ordered pairs, and algebraic representations, because these three types of representations are the most common in the secondary schools' curriculum in Kurdistan. The study investigates secondary school students' understanding of the concept of function in the Kurdistan Region of Iraq. This study is limited to focus only on secondary school students in Erbil city, the capital of the Kurdistan Region. To carry out the investigation, a questionnaire was administered in secondary schools in Erbil city.

This study investigates secondary school students' understanding of the concept of function in Kurdistan, with the aim of disseminating the results and reporting them to the Kurdistan Regional Government (KRG) Ministry of Education so as to contribute to improving the teaching of Mathematics in Kurdistan's secondary schools.

The study is guided by the following main research questions:

(1) What is the students' understanding of the definition of function?

(2) Can students recognize different representations of function?

(3) Can students convert between diverse modes of representation?

Theoretical background

The concept of function

Many studies have investigated students' understanding of the concept of function, concentrating on testing students' ability in recognizing different representations of function and the transition between them (Kurt & Cakiroglu, 2009; Gagatsis & Shiakalli, 2004; Schwarz et al., 1990; Sajka, 2003; Hitt, 1998).

Mathematics educator researchers have proposed different explanations for students' difficulties in understanding the concept of function. Firstly, the dual nature of function creates problems. The concept of function is multifaceted. It can be explained as a conceptual understanding; for example, a function is a set of ordered pairs. It can also be interpreted as a procedural understanding; for example, a function is a process for accessing one system from another (Sfard, 1991; Skemp, 2012). Secondly, Vinner and Dreyfus (1989) point out that the majority of students are unfamiliar with the terminology related to the understanding of the concept of function. For instance, students are not familiar with the symbol $f(x)$. Finally, according to Schwarz et al. (1990), students solve mathematical problems mechanically rather than meaningfully. In other words, students solve mathematical problems by routine, they can apply the mathematics rules and follow the procedures to solve a problem without understanding the core of the steps.

Mathematics educators agree that the concept of function is fundamental to learning mathematics, especially at the secondary level (Kalchman & Case, 1998). There is a strong relationship between mathematics achievement and having conceptual knowledge. Students who have good experience in mathematical concepts tend to achieve a good grade in Mathematics as a subject (Zulnaidi & Zakaria, 2010).

The following section reviews the knowledge base regarding students' understanding of three main aspects: understanding the definition of a function, recognition of multiple representations of the same function, and conversions between different representations of the function.

The theoretical perspective used in this study is Duval's (2006, p. 2) approach, which "considers the conversion of representations of the same mathematical situation as a fundamental process leading to the understanding of the particular situation". It is underpinned by the assumption that students' ability to define a function, their ability to recognize a function in different modes of representation, and conversion between them, can show their understanding of the concept of function.

Understanding the definition of function

A study by Tall and Vinner (1981) shows the relationship between the concept image, defined as "a set of properties associated with the concept... together with the mental picture" (Tall & Vinner, 1981, p. 293), and the concept definition, the "verbal definition that accurately explains the concept in a non-circular way" (Tall & Vinner, 1981, p. 293). Responses from college students to a questionnaire revealed that students depended on the concept image to explain the concept definition of a function. The participants were given five different functions. Each of them was represented in algebraic and graphic form. The question was: *Which of these functions represent a continuous function? Explain the reason for your answer.* Most of the students used the concept image to solve the problem instead of using the concept definition of continuous function. They selected the functions that did not have a gap in the graph as a continuous function. Vinner and Dreyfus (1989) in their research found that the participants answered 'the definition of function' task in different ways. The students' interpretation of function was categorized into seven types: correspondence, dependence, rule, operation, formula, representation, and others. While the formal definition of the function was given to them at school, most of them did not use this definition when they were asked about it. However, students' responses depended on the concept image. As a result, it would seem that students had problems with both the concept image and the concept definition because they could not recognize either concept properly (Vinner & Dreyfus, 1989).

Multiple representations of function

Many researchers have confirmed that different representations of function provide students with the best chance of building a conceptual understanding of function (Tall & Vinner, 1981; Hitt, 1998; Markovits et al., 1986; Ainsworth et al., 2002; Greeno & Hall, 1997). According to Duval's (1999) theoretical perspective, there are many types of representation of the function, each of them representing a different side of the concept. Yet none of them alone can describe the idea of function completely. All these studies show that students need to recognize different representations of function to help them to deepen their understanding of the concept of function (Kurt & Cakiroglu, 2009).

On the other hand, many studies have shown that students have a problem in identifying different representations of the same function. Markovits et al. (1986) found that students face difficulties and confusion with functions that are represented by a set of ordered points. Furthermore, students have problems in coordinating between different representations of functions such as graphs, tables, and equations (Kurt & Cakiroglu, 2009; Schoenfeld et al., 2019). Also, Sfard (1992) shows how students were unable to relate algebraic representations with graphical representations of the function. In contrast, Dubinsky and Wilson's study (2013) illustrates that with an appropriate curriculum and method of teaching, students do have the ability to absorb different representations of function and convert between them. In their study, after seven weeks of being taught in the program, participants scored highly in recognizing diverse modes of representation of the function. This shows that pedagogy and the curriculum affect students' success in identifying different representations of the function.

Conversion between different representations of function

According to the outcomes of previous studies, students' ability to make the transition between different representations of function is important, see for example, (Dufour-Janvier et al., 1987; Even, 1993). Researchers have different views about the difficulties that students face when required to convert a function from one representation to another, given the complexity of the process (Elia, Panaoura, et al., 2007; Yerushalmy, 1997). Furthermore, students in different classes perform differently in recognizing and converting between representations of the function. In studies by Elia, Gagatsis, and Demetriou (2007), and Kurt and Cakiroglu (2009), students in a higher class or grade performed better in both recognizing and converting between different representations of the function.

Some conversions between diverse modes of representation of a function are more difficult than others. The transition from a graph is more complicated than the change to the graph (Elia, Gagatsis, & Demetriou, 2007). Also, a transformation that contains symbols is more difficult for students than a shift that contains written language (Hitt, 1998; Lesh et al., 1987). Other researchers, on the other hand, focus on the crucial role that the curriculum and pedagogy have in improving students' conceptual understanding of function (Dubinsky & Wilson, 2013; Gagatsis & Shiakalli, 2004; Kurt & Cakiroglu, 2009). For example, Dubinsky and Wilson (2013) show that students can convert a function from one representation to another if they have an appropriate curriculum, and a suitable method of teaching is used.

To sum up, previous research suggests that there are three main ways in which students can be tested to indicate their understanding of the concept of function. Firstly, students depend on the formal definition of the function rather than the concept image (Vinner & Dreyfus, 1989). Secondly, students' ability to recognize different representations of the function indicates their understanding of the notion of function. According to Duval's theory (1999), each type of representation describes one aspect of function and the variety of representations in a mathematical situation gives the whole idea of the notion. Finally, students' ability to convert a function from one type of representation to another plays a crucial role in their understanding of the concept of function. There is also a strong relationship between students' ability to problem-solve and their ability to make transitions between representations (Gagatsis & Shiakalli, 2004). Therefore, the present study tests three main aspects of students' ability: understanding the definition of function; recognizing different representations of function and converting functions between diverse modes of representation.

Methodology

Participants

The participants of this study were 366 students in grade eight (14 years old). Twelve secondary public schools were chosen in Erbil city in the Kurdistan Region of Iraq. The participants were chosen based on some basic criteria: a range of different geographical locations in the city to cover a range of socio-economic backgrounds; grade eight was chosen because in this grade students have to have a concept image of function; and consent was required from the head of the school, students and students' parents. 342 out of 366 students (93.4%) completed the test.

Research ethics

In this study, the principles of research ethics were carefully considered. The number of the ELTE PPK Research Ethics Committee license is 2020/209. All the names of schools and students that participated have been anonymized. Initial checks ensured that the questions were unambiguous and that the time allocated (40 minutes) was enough for students to answer all the items.

Measure

The test was designed to investigate three main aspects of students' understanding: students' knowledge of the definition of the function; students' ability to recognize different representations of function; and students' ability to convert a function from one type of representation to another.

According to the literature, students' understanding of the concept of function can be investigated by focusing on their ability to manipulate different representations of function and the transition between them (Elia, Gagatsis, & Demetriou, 2007; Schwarz et al., 1990; Sajka, 2003; Hitt, 1998). The first three tasks tested students' ability to understand the definition of function. The next three tasks (Task 4–6) tested students' ability to recognize different representations of the function and different symbols of the function. The last three tasks (Task 7–9) tested students' ability to convert a function between diverse modes of representation. In the first six tasks (Task 1–6), respondents were asked to justify their answers in order to provide more detailed information and increase data accuracy. The test is presented in the Appendix.

Procedures

The first author visited Kurdistan of Iraq to administer the questionnaire in twelve selected public secondary schools in Erbil city, which is the capital of Kurdistan. After getting consent in the twelve secondary schools, the test was administered during one lesson period in the school.

Analysis

Each of the nine tasks in the questionnaire was analyzed separately and responses were categorized according to the nature of the task. In the first task, respondents were asked to define the function. Responses were then categorized as *correct definition, incorrect definition, or ambiguous definition.* As for the consecutive tasks (Task 2–9), responses were categorized as *correct, incorrect, or no response given.* The justifications required for Tasks 2–6 were categorized as *correct, incorrect, or none given.* For each task, descriptive statistical indices were calculated.

Results

According to Check and Schutt (2012, p. 279), "the first step in data analysis is usually to describe the distribution of each variable". In this study, analysis began with the participants' responses where each task was coded to facilitate the analysis process (Johnson & Christensen, 2019).

First Task

- Note f(x) is a function from $R \to R$
  Can you give the definition of function?

In this task, students were asked to define the function. Students' responses were categorized as: *correct, incorrect, or ambiguous.* Incorrect definitions did not have a relation with the meaning of the concept of function, whereas responses that only gave a partial definition were categorized as ambiguous. Out of the 100% of students who responded to this specific task, only 13.4% provided the correct definition. In other words, the majority had difficulties in giving the proper interpretation of the function, with 24.7% of students providing an incorrect definition and 62% providing an ambiguous definition (see Figure 1).

Correct definition	A function is a pairing between two sets of numbers in which each element of the first set is paired with exactly one element of the second set (Xoshnaw & Muhsen, 2015).
Incorrect definitions	"Function is a relation between two different sets, for each x has to have correspondence with y". "Function is a relation between two sets; one of them is called the domain, and another is co-domain, one of them is dependent, and another is independent".
An ambiguous or partially correct definition	"The function is a relation between two things such as x and y." "Anything that contains x and y, and represented by a straight line is called function."

Table 1. Some extracts from the students' answers

Their justifications indicate that students did not depend on the concept definition of the function but on a picture of the function that they had in their mind, which is called a concept image of the function. Out of the 342 participants, 45 students (13.2%) gave a correct answer, the majority of them (62.0%) provided a partially good solution, while 24.9% of them gave an incorrect or missing answer.

Second Task

- Examine which of the following correspondences are functions?

This task contained four diagrams and requested that students identify whether each one represented a function or not. Respondents were also asked to justify their answers. It can be seen in Table 2 that the majority provided the right answer in all sections (see the Appendix for the layout of the tasks).

All four items were yes-or-no items with an open-ended question on justification. Taking account of the guessing chance limit for the yes-or-no question (even though there was a third possibility 'I do not know' provided), item C) proved to be relatively difficult, while the others were solved relatively well. The rate of correct justifications was 10–25% below the achievement on the yes-or-no item.

Task	*Correct solutions (%)*
2/A	77.8
2/A justification	58.3
2/B	71.9
2/B justification	61.0
2/C	52.3
2/C justification	33.0
2/D	80.1
2/D justification	55.8

Table 2. The percentage of students' answers for the second task

Third Task

- Every relation between two sets is a function.

In this task, students were asked a question to reveal their understanding of the definition of the function. Students' answers were classified as follows: *correct, incorrect, and no answer given.*

More than half of the students (68.1%) provided the correct solution in this task, although only 43.8% of them provided the correct justification. Examples of incorrect justifications include: *Because each set is different; Maybe there is a problem in one of these sets.*

These types of comments provide an insight with regards to the nature of the students' misunderstanding of this question, even if they chose the correct answer.

Fourth Task

- Are $(f(x) = x^2)$ and $(y = x^2)$ the same function?

In this task, respondents were asked about different symbols of the function. The rate of correct answers was around the mere guessing chance. For example: *This is not a function because there is no f in it; We cannot call it function without* $f(x)$.

These comments show that the students have misunderstood the symbol $f(x)$, and they do not comprehend that $f(x)$ is only a symbol that can be easily replaced by any other symbol or letters instead of $f(x)$.

Fifth Task

- Do these graphs represent a function?

This task contained two questions (Question A and Question B, see the Appendix). A graph was provided for each question, and students were asked to decide whether these graphs represent a function or not. Item A) depicted a drawing obviously not that of a function, and item B) contained a pictorial representation of a function. For item A), approximately half of them could not provide the right answer.

More than half of the participants answered Questions A and B correctly, 67.5 and 59.9, respectively. Justification of their answer, however, proved to be a challenge with 38.1 and 46.3% provided correct justifications, respectively.

Sample answers are as follows: for Question A, *This is not a function because if we draw a vertical line on this graph, it intersects the graph at more than one point*; for Question B, *This is a function because two numbers in x correspond to one number in y.*

It seems that these students have a problem with graph representation and perhaps lack information about this type of representation.

Sixth Task

- Do these ordered pairs represent a function $\{(2,5),(2,6),(3,7)\}$?

This task tested students' recognition of function in an ordered pair representation. Students were given some ordered pairs and then asked to identify whether these ordered pairs represented a function or not.

This task revealed that students have a problem with the representation of the function in ordered pairs, since 55.3% gave a correct answer. Students do not appear to be entirely familiar with this type of representation. Another issue worth mentioning is that some participants graphed the ordered pairs and then decided on their answers. The majority of these participants made mistakes because they could not find a relationship between the graph of the ordered pairs and the definition of the function.

A common answer was: *This is a function because the graph of these ordered pairs is curving.*

Furthermore, not all students who answered this task correctly could provide a correct justification: 58.7% of those who provided correct answers also provided correct explanations. Some of the responses show a lack of familiarity with this type of representation, even when providing a correct justification.

Seventh Task

- Which one of these graphs represents the graph of the following functions?

This task, containing two questions (Questions A and B, see the Appendix), assessed students' ability in converting functions from an algebraic to a graphic representation. Each item provided the participants with a function in algebraic representation, and the choice of the corresponding visual representation of the function was required.

A higher percentage of students provided the right answer to Question B than to Question A, 66.1% and 53.2%, respectively. The function in Question A was $f(x) = x$, and one of the three choices was the graph of the parabola. The trouble is that 35% of the participants (which is the rate around the guessing chance) chose the graph of the parabola as a graph of the function. Therefore, this task revealed that students did not realize that $f(x) = x$ is a linear function, and they did not understand that the graph of the first-degree function is always a straight line, which shows the students lack in understanding the concept of the function.

Eighth Task

- Find the expression of the function represented by these graphs.

This task is the exact opposite of the previous task and sought to assess students' ability to convert a function from a graphic representation to an algebraic representation. This task comprised Questions A and B, each involving a function provided as a graph representation; students were asked to find an algebraic representation for the graph. 46.5% of participants provided a correct answer for Question A, while only 25.7% answered Question B correctly. Since this is below the mere guessing chance in a closed-question format item providing three options only, this result indicates that students find this transition particularly problematic. And the most popular mistake was that students could not differentiate between the graph of $f(x) = x - 3$ and the graph of $f(x) = x + 3$ because they do not understand these functions conceptually.

Ninth Task

- Find the graph of the function that represented by ordered pairs $\{(1, 2), (2, 5), (3, 10), (4, 17), \ldots\}$?

In this task, participants were asked to choose an appropriate graph for a function that was represented by ordered pairs. This task aimed to discover students' ability to convert a function from an ordered pair representation to a graph representation.

Approximately half of the participants could not answer this question correctly (53.2%). This result highlights the difficulties students have in converting a function from an ordered pair representation to a graph representation.

Connections between students' answers on different tasks

The test contained altogether 24 items yielding an appropriate reliability (Cronbach's alpha = .85), therefore there is a good internal consistency of the answers. The tasks measured different knowledge constructs within the realm of functions, consequently cluster analysis can provide information on what kinds of item sets proved to assess similar psychological characteristics. Figure 1 presents the dendrogram obtained from cluster analysis with the method of the furthest neighbour clustering and using Pearson-correlations.

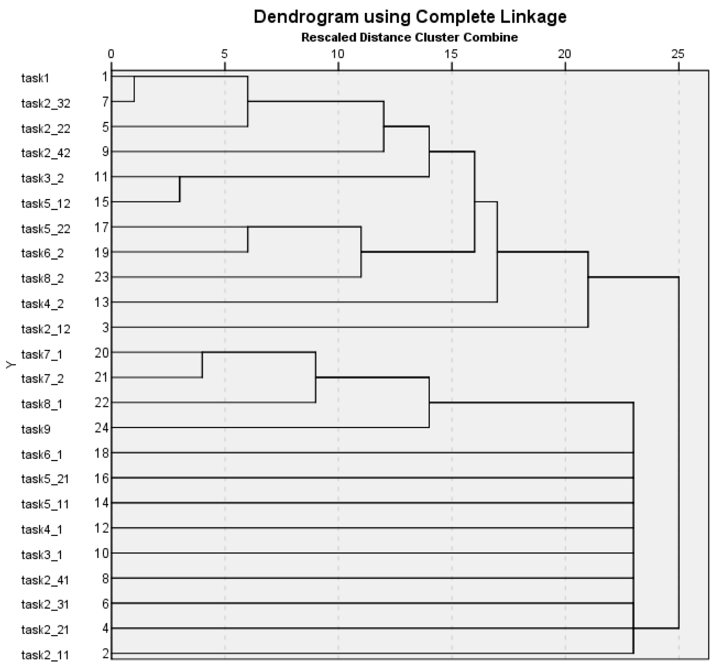


Figure 1. Dendrogram on the items of the test

Figure 1 presents two main clusters of the test variables. The upper set contains primarily items requiring justifications. This set includes also the first, trichotomic item. Importantly, the items on conversion between different forms of function representations belong to one similarity group of variables, thus indicating that albeit the rate of correct answers differed quite much from task to task, there is a strong correlation between these items. Nevertheless, the dendrogram nicely indicates that a mere verbatim knowledge on the definition of function (Task 1) is neither sufficient nor necessary for obtaining correct answers on conversion-type tasks.

As for the relative weight and importance of different pieces of knowledge, we conducted regression analysis in order to choose those items that make the most remarkable contribution to the differences between students' overall achievement. Stepwise regression analysis showed that as few as four items can explain 95% of the variance of the test score. These items are: Task 1, Task 7/B, Task 8/B, and Task 4 justification. Although belonging to different item clusters, these four items together would assess students' knowledge quite precisely if we had only a short time to test their knowledge on function. Two of these crucial items require conceptual knowledge of functions, and two of them tested the ability of conversion between different forms of representations. Consequently, the overall knowledge level on function relies on both verbal and pictorial representations and knowledge.

Discussion

Students' understanding of the definition of function

This study has revealed students' difficulties in providing a proper definition for the concept of function in that the majority of them provided an ambiguous definition. This suggests that although students may have some idea about the definition of the function, it is incomplete. As a result, students defined the concept of function based on the concept image instead of the concept definition, or they only provided a partial definition.

The students' responses to the second task demonstrate a better understanding of the definition of the function than in the first task. It seems that respondents were able to represent the definition of the function by a correspondence of sets.

The majority of students who answered incorrectly in the second task in section C provided the following justifications: *It represents a function because* *the elements in the domain have only one correspondence to elements in the co-domain; None of the elements in the domain has a relation with more than one element in the co-domain.*

These answers indicate that the students concentrated on the part of the definition of function and they forgot to think about another part of it, in which all elements in the domain have to have a correspondence with elements in the co-domain. Students' responses to this task also suggest that mathematics teachers are likely to use different expressions to explain the relation between domain and co-domain in a function. For example, mathematics teachers might use the relation between a father and his children to explain a function to students, as the following justification illustrates: *A child can't have more than one father, but the father can have more than one child.*

It seems that "child" represents the element in the set of domain and "father" represents the element in the set of co-domain.

Results from the first three tasks in the questionnaire revealed difficulties with giving a proper interpretation of the function, and evidence that this definition was not fully understood. This result is consistent with the findings of Elia, Gagatsis, and Demetriou (2007) where the majority of students provided ambiguous interpretations, and just 35% were able to provide an accurate description.

Students' ability to recognise different representations of function

Task 4 in the questionnaire was designed to test students' ability to recognize different notation of the same function. While Tasks 5 and 6 were designed to test students' ability to recognize different representations of the function.

The findings revealed three main difficulties for students in this area. Firstly, students had difficulties with the symbol sense and interpretation of symbol $f(x)$. Secondly, students had a problem with finding a relationship between the definition of the function and the graph representation. Finally, students were not entirely familiar with ordered pair representations.

These findings are consistent with the conclusions of previous studies: in a study by Vinner and Dreyfus (1989), students had a problem with terminology which relates to an understanding of the concept of function; students do not comprehend that $f(x)$ is only a symbol that can be easily replaced by any other symbol, according to Dreyfus and Eisenberg (1982) students have difficulties with representing the symbol $f(x)$ and misconceptions of $f(x)$ as a notion; in studies by Kurt and Cakiroglu (2009), and Schoenfeld et al. (2019), students had difficulties coordinating between different representations of functions such as graphs, tables, and equations; Markovits et al. (1986) found that students faced difficulties and confusion when presented with functions that are represented by ordered pairs.

Another problem with graph representation that arose is that respondents could not find a relation between the definition of the function and the graph representation. For example, one student wrote: *"This graph does not represent a function because it is not a straight line"*.

As in previous studies, see for example (Kurt & Cakiroglu, 2009), this task reveals that students faced difficulty with the graphical representation of the function. Kurt and Cakiroglu (2009) also found that the majority of participants did not have enough ability to understand and recognize different representations of the function. The researchers pointed out that different representations are not considered necessary in the school curriculum and argued that this is the main reason why students have a problem and are unable to recognize different representations of the function, specifically, graph representation (Kurt & Cakiroglu, 2009).

On the other hand, participants in the present study seem to be more comfortable with graphical representation. This result is consistent with findings in Elia, Panaoura, et al.'s (2007) study, in which the majority of participants recognized the graph of the function. In other words, participants performed better in graphical representation compared with other representations. This finding is in stark contrast with Kurt and Cakiroglu (2009) who found that graphical representation was the most challenging for the students.

Students' ability to convert a function between diverse modes of representation

The last three tasks in the questionnaire focused on students' ability to transform a function from one type of representation to another. According to the findings, participants had difficulties in conversion between different representations of the function, with the most challenging being the conversion from the graphic representation of a linear function that exists in their curriculum to an algebraic representation.

This finding is in line with Sfard's (1992) study in which students were unable to make a bridge between algebraic representation and graphical representation of the same function; also with that of Artigue (1992) and Kerslake (1986), indicating that mathematics teachers and university students in mathematics departments faced problems with conversion between different representations of functions; according to Gagatsis and Shiakalli's (2004) study, students found some transitions easier than others. The transition to the graph was more comfortable than the transition from the graph, and the most difficult conversion was found to be the transition from graphical representation to verbal and algebraic representation. In addition to this, Kaldrimidou and Ikonomou (1992) found that Greek students sidestepped the use and interpretation of graphical representation, preferring instead to use algebraic representation. In contrast, in Hitt's (1998) study, which was conducted with mathematics teachers, the highest rate of correct responses was in the transition function from the graphic representation to other representations.

Relation between the three perspectives

In this study, the majority of the participants who provided the correct definition of a function and understood it, that is those who provided correct answer for questions in Task 1, 2 and 3, also performed well in recognizing different representations of a function and conversion between them. 78.5% of those who provided a correct definition of function and understood it, were also able to provide the right answer when it came to recognizing different representation tasks in the questionnaire. Also, 71.4% of the participants who provided a correct definition of function, were able to respond correctly when it came to converting a function between diverse representation tasks in the questionnaire.

These findings strongly suggest a positive relationship between the ability to define a function and understanding it and the ability to recognize different modes of representation of the function and converting between them.

Conclusions and educational implications

This study aimed to investigate students' understanding of the concept of function by focusing on three main aspects: students' knowledge of the definition of the function; students' ability to recognize different representations of function; and students' ability to convert a function from one type of representation to another. The study revealed that students have difficulties in providing a proper definition of function and understanding it. Three main problems were identified in students' responses to the questionnaire. Firstly, students confused concept image with a concept definition of the function in that they defined a function based on a concept image rather than a concept definition. Secondly, most students provided a partial definition of function and neglected the other part, resulting in ambiguous definitions. Finally, many students who answered the tasks correctly, demonstrating the students' understanding of the definition of the function, could not justify their answers adequately. This suggests that students have misconceptions with regards to the definition of the function.

As the results of this study show, students have difficulty in recognizing different representations of function. Two main reasons have been identified for this difficulty. Firstly, students had a problem with symbol sense. Secondly, students had difficulties in finding a relationship between the definition of a function and different representations of it. However, it seems that the students are more comfortable with graphical representation because, in this study, the students' performance in the graphical representation task was better than in other tasks. As a consequence, it can be asserted that the students did not grasp a wide range of aspects when identifying different representations of the function.

According to the results, students had difficulties in the conversion function between different modes of representation, the most difficult being the conversion function from graphic representation to algebraic representation. The findings in this study have proved that the students' ability to provide a correct definition of the function has a positive relationship with their ability to recognize different representations of function and conversion between them.

Based on the findings, secondary school students in Kurdistan have difficulty in providing a proper definition of the function, recognizing diverse modes of representation of functions, and converting between different representations of the function. Therefore, it is recommended that educators concentrate more on students' understanding of the concept of function, because function is one of the fundamental concepts in mathematics learning in that it has a relationship with other mathematic subjects.

In terms of further research, mathematics researchers could identify more reasons behind students' difficulty with the concept of function by conducting interviews with both mathematics teachers and students. More insights into students' problems with this concept could be generated by presenting students with some function-related mathematical problems in the context of an interview, and then asking them to solve these problems using a talk aloud technique. Another possible study could be based on observation of Mathematics classes, in order to gain more information regarding the methods used in teaching mathematics, specifically in teaching the concept of function.

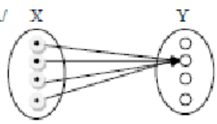
Appendix

Questionnaire

Note / f (x) is a function from R ⟶ R

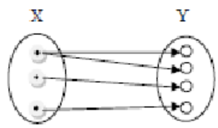
Task 1/ Can you give the definition of function?

Task 2/ Examine which of the following correspondences are function?

A/


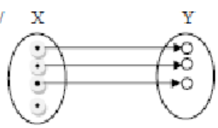
Yes ☐ No ☐ I do not know ☐

Justify your answer: -----------------------------

B/


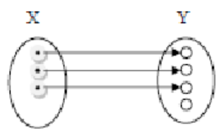
Yes ☐ No ☐ I do not know ☐

Justify your answer: -----------------------------

C/


Yes ☐ No ☐ I do not know ☐

Justify your answer: -----------------------------

D/


Yes ☐ No ☐ I do not know ☐

Justify your answer: -----------------------------

Task 3/ Every relation between two sets is a function. Yes ☐ No ☐ I do not know ☐

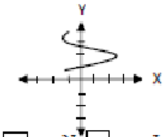
Justify your answer: --

Task 4/ Are ($f(x) = x^2$) and ($y = x^2$) the same function? Yes ☐ No ☐ I do not know ☐

Justify your answer: --

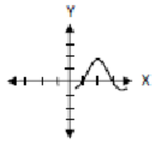
Task 5/ Do these graphs represent a function?

A/



Yes ☐ No ☐ I do not know ☐

Justify your answer: ------------------------------



Yes ☐ No ☐ I do not know ☐

Justify your answer: ------------------------------

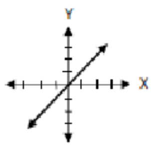
Task 6/ Do these order pairs represent a function {(2,5), (2,6) , (3,7) }?

Yes ☐ No ☐ I do not know ☐

Justify your answer: --

Task 7/ which one of these graphs represents the graph of follow functions?

A/ $f(x) = x$



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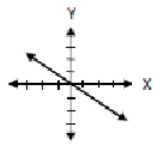

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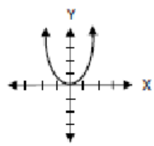

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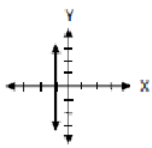
B/ $f(x) = 1$



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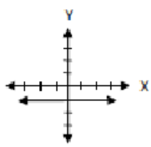

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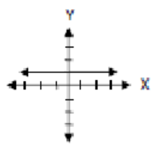

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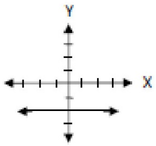
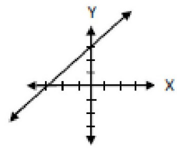
Task 8/ Find the expression of the function represented by these graphs?

A/

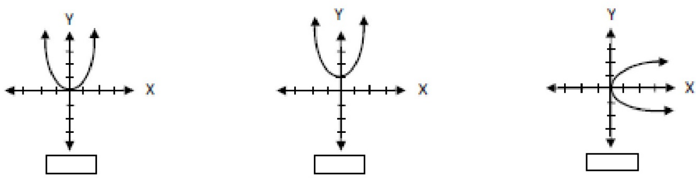


B/



$f(x)=x-2$ ☐ $f(x)=-2$ ☐ $f(x)=x^2$ ☐ $f(x)=3$ ☐ $f(x)=x-3$ ☐ $f(x)=x+3$ ☐

Task 9/ find the graph of the function that represented by order pair {..., (1,2), (2,5), (3,10), (4,17), ...}.



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