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Straight line or line segment? Students' concepts and their thought processes

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Abstract. The article focuses on students' understanding of the concept of a straight line. Attention is paid to whether students of various ages work with only part of a straight line shown or if they are aware that it can be extended. The presented results were obtained by a qualitative analysis of tests given to nearly 1,500 Czech students. The paper introduces the statistics of students' solutions, and discusses the students' thought processes. The results show that most of the tested students, even after completing upper secondary school, are not aware that a straight line can be extended. Finally, we present some recommendations for fostering the appropriate concept of a straight line in mathematics teaching.

Key words and phrases: straight line, line segment, students' thoughts, teaching geometry.

MSC Subject Classification: 97C30, 97D70, 97G40.

Introduction

A basic understanding of geometric concepts is important for the development of students' thinking and geometric imagination, both of which facilitate their progress in mathematics. In our long-term research, we deal with the understanding of various geometrical concepts among Czech students. As a part of this research, we assigned three tests. Two tasks of the tests were related to the infinity of a straight line.

Mathematicians distinguish between potential and actual infinity. While the first one is found already in the Aristotelian conception in the 4th century BC, the second approach was fully elaborated by Georg Cantor in the 20th century (Moreno & Waldegg, 1991; Fischbein, 2001). Students encounter the term *infinity* before they can fully understand it. However, based on our experience, we know that it triggers curiosity and interest in them. We consider the right intuitive perception of a straight line as an infinite object to be important for students. Therefore, we are concerned with the question: *Do pupils and students only work with the part of the straight line shown, or do they realize that it can be extended?*

Theoretical background

A straight line is one of the fundamental concepts of Euclidean geometry. Euclid understood a straight line as a line segment which can be extended repeatedly. According to the Czech national curriculum for elementary education, pupils should know the terms *straight line* and *line segment* no later than the end of the 5th grade. However, as early as the 3rd grade, the recommended outcome is that “a pupil recognizes a straight line and a line segment, draws them and knows how they are named” (MŠMT, 2017: 33). In alignment with this recommendation, a straight line is introduced in most contemporary Czech mathematics textbooks in the 2nd or 3rd grade. A line segment is usually taught earlier.

In Czech textbooks for primary schools, the difference between a straight line and a line segment is usually described as follows: “we cannot measure a straight line” or “we cannot draw a whole straight line; it is unlimited”. An emphasis is often placed on how to name a straight line. We find the advice “to draw straight lines from one edge to the other edge of a given area (paper, blackboard, etc.)” in the guidelines for teachers (Divíšek, Hošpesová & Kuřina, 2000). This topic is not usually discussed in detail in mathematics textbooks for secondary schools. Upper secondary school textbooks only introduce that a straight line is determined by two points. They do not pay much attention to its infinity.

Before starting our testing, we asked two groups of students of different ages to answer "What is a straight line, in your opinion?" Most students used the term infinity to describe a straight line; there were no significant differences between the groups (Robová et al., 2019). According to Fischbein, Tirosh and Hess (1979) the infinity intuition is relatively stable at approximately 12–13 years of age.

Monaghan (2001: 244) found that upper secondary school "students' primary focus on infinity was a process, something which goes on and on." On the other hand, if students talk about "going towards infinity" (about numbers, straight lines etc.), it can refer to actual infinity. Hannula et al. (2006) was focused on students in grades 5 and 7. The authors distinguished three levels of students' understanding of infinity: no understanding, understanding of potential infinity and understanding of actual infinity. They found that potential infinity was understood earlier than the actual one, and, in general, "students have no clue of infinity" (Hannula et al., 2006: 333).

Jirotková and Litter researched into concepts of a straight line in Czech and English primary school pre-service teachers (Jirotková & Litter, 2003), and in Czech and English lower secondary school students (Jirotková & Litter, 2004). They found an occurrence of three students' ideas about a straight line. Most respondents preferred the idea which refers to two infinities on a straight line. According to another research into the problem of students' conceptions of the infinity of numbers through interviews with 5th grade students, found that "students' potential of epistemological thinking is surprisingly high in grade 5" (Boero, Douek & Garuti, 2003: 128).

Methodology

We prepared three written tests focused on basic geometrical concepts, with respect to the Czech national curriculum: Test I for pupils finishing primary school, Test II for students finishing lower secondary school, and Test III for students finishing upper secondary school. Each of these tests was given to a small group of students in the respective age categories as a pre-test for the purpose of verifying the comprehensibility of the tasks and instructions, as well as to determine the time limit. The pre-test also included several semi-structured interviews between one of the researchers and one or two students. Students were asked to clarify their solutions of the given tasks.

The tests were modified on the basis of the pre-test outcomes and then administered to 1,414 Czech students (Test I: 505 students; Test II: 437 students; Test III: 472 students) who were about to move on to a higher level of education, as well as to 44 pre-service mathematics teachers in the last two years of their university studies. The participants were selected on the basis of their availability. The tests were solved anonymously, students were differentiated only by gender. All tests were personally administered by one of the researchers. Students were encouraged to read the assignment carefully.

Test I included, inter alia, two tasks related to the infinity of a straight line. The question in the first one (Fig. 1 on the left) was: "Do the straight lines *a* and *b* have an intersection point?" This question was formulated as a closed question; the possible answers were: "YES", "NO" and "NO OPINION". The question in Task 2 (Fig. 1 on the right) was: "How many common points do the circle *k* and the straight line *p* have?" The same task was also included in Tests II and III. This question was open in Test I and III, students were asked to write a number of common points. In Test II, this question was closed; possible answers were: 0, 1, 2, 3, more than 3. In addition, students could write any comment for each task.

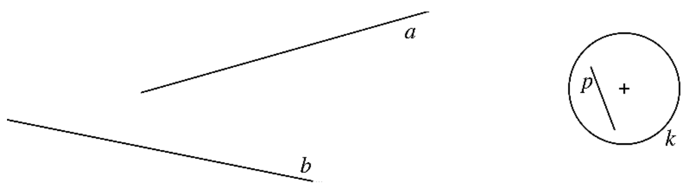

Figure 1. Tasks on an extension of the part of the straight line shown

The completed tests were assessed qualitatively. For this purpose, the tests were divided into several groups. Each student's answer for each task was assigned a code. In Task 1, we used the code *Y* for the answer "YES", *N* for the answer "NO", and *N-O* for the answer "NO OPINION". In Task 2, the answers were coded with the appropriate number (in Test II, we used the code *>3* for "more than 3") and the code *inf* was used for the answer "infinity". The tests from every group were coded independently by different pairs of researchers. Any discrepancy in the coding of a specific student's answer was discussed among the whole research team until a consensus was reached. From the data, the absolute and relative frequencies of the codes were determined and some relationships were observed and processed.

Results

The results of Tasks 1 and 2 are presented in Tables 1–4, where, except for total numbers of particular answers, the distribution of answers by gender is given.

Test I, Task 1	*Y*	*N*	*N-O*	*OA*
Male	48.9%	48.0%	2.2%	0.9%
Female	41.1%	54.6%	3.2%	1.1%
Total	44.5%	51.7%	2.8%	1.0%

Table 1. Results of Task 1; code *OA* (other answer) includes missing answer and multiple choice options

More than 50% of pupils gave the answer “NO” and about 45% of pupils chose the “YES” option in Task 1. The difference between male and female is evident in Table 1; however, the chi-squared test did not confirm the dependence of the “YES” option on gender. Only about 3% of all respondents admitted they did not know.

Test I, Task 2	*2*	*0*	*1*	*OA*
Male	24.4%	68.0%	1.8%	5.8%
Female	16.8%	76.8%	1.8%	4.6%
Total	20.2%	72.9%	1.8%	5.2%

Table 2. Results of Task 2 in Test I; code *OA* (other answer) includes missing answer and answers with absolute frequency less than 3

Test II, Task 2	*2*	*0*	*>3*	*OA*
Male	21.6%	75.8%	0.9%	1.7%
Female	15.5%	80.6%	1.5%	2.4%
Total	18.8%	78.0%	1.1%	2.1%

Table 3. Results of Task 2 in Test II; code *OA* (other answer) includes missing answer and rarely occurring answer “1”

In Tests I and II, about 20% of students chose the answer “2”, and more than 70% gave the answer “0” in Task 2. On the other hand, in Test III, 40% of students chose the answer “2”, and about 56% chose “0”. The answer “2” was given most frequently by pre-service teachers (P-ST), see Table 4. It is worth mentioning an occurrence of the answer “1” in Test I, and “more than 3” or “infinity” in Tests II and III. We can observe differences between males and females in all the tests, but only in Test III does the chi-squared test confirm the dependence of the “2” option on gender.

Test III, Task 3	*2*	*0*	*inf*	*OA*
Male	49.1%	47.4%	2.0%	1.5%
Female	28.4%	68.0%	2.7%	0.9%
Total	40.1%	56.4%	2.3%	1.2%
P-ST	45.5%	50.0%	4.5%	0.0%

Table 4. Results of Task 2 in Test III; code *OA* (other answer) includes missing answer and answers with absolute frequency less than 2

We were interested in the existence of a dependence between the occurrence of answers “YES” in Task 1 and “2” in Task 2 in Test I. According to the chi-squared test, the null hypothesis that the occurrence of these answers is not related was rejected. On the contrary, the results showed that a significant connection exists.

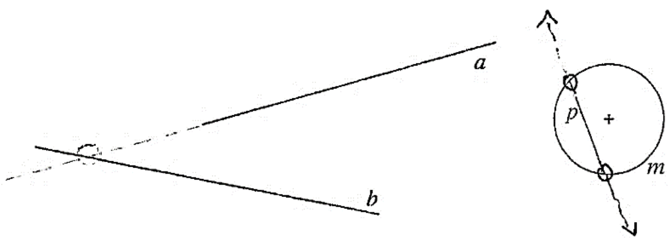

Figure 2. Examples of students’ solutions

Several students extended the given line in the picture to demonstrate the existence of intersection points (Fig. 2). Some students from each of the age-differentiated groups added a brief clarification and comment on their choice. All of the comments can be summarized as follows.

Task 1: 1A – Yes, the straight lines are infinite. 1B – Yes, but not in the picture. 1C – Yes, when it is extended. 1D – No, not until I draw it.

Task 2: 2A – The straight line has no end or beginning. 2B – The straight line is infinite. 2C – There are 2 common points with the straight line, but 0 with the line segment. 2D – It is a straight line. Why is it not extended? 2E – This is not a straight line!

Discussion and Conclusion

Geometrical terms like a straight line, a line segment, etc., are abstractions; we can only visualize them using their models which are based on real-life experience. However, students cannot have real-life experience with infinity. Therefore, they must create a tacit model of infinity (Fishbein, 2001), that is a model created solely in their minds. A very important factor is also the context in which a student encounters the concept of infinity (Monaghan, 2001). The term *straight line* was clearly stated in the word problems of both presented tasks. Therefore, it is necessary to consider the infinity of given straight lines; hence, from the mathematical point of view, the ideal answers are “YES” in Task 1 and “2” in Task 2. The problem is that a straight line must be always represented as a line segment. We intentionally drew the line segments shorter than usual in our tasks and observed how the students handled it.

Despite this unusual image of a straight line, about half of the tested pupils realized that they could extend the line and that the given straight lines have an intersection in Task 1 in Test I. Task 2, in which the part of the straight line was drawn in the inner area of the circle, was evidently more confusing. In Tests I and II, only about 20% of respondents answered that there are 2 common points. However, this response occurred in 40% of students in Test III. Thus, some improvement is shown with increasing age. The chi-squared independence test checking the relationship between Tasks 1 and 2 in Test I proved that pupils were consistent in their opinions (most of those who chose “NO” in Task 1, answered “0” in Task 2). However, Tables 1 and 2 show that Task 2 was more confusing then Task I. The answers which were in accordance with the perception of a straight line as an infinite object were given more frequently by males. The same conclusions were also reached by Hannula et al. (2006). Nevertheless, a statistically significant sex difference in favour of males was observed only in Test III.

Several pupils gave a surprising answer of “1” in Task 2 in Test I. Based on other tasks, we know that some pupils perceive the whole line segment as a single point (Robová et al, 2019). They probably thought that the pictured segment line was equal to the intersection of given objects, thus they considered the inner area of the circle (disc) as a part of the circle. Moreover, they do not have a proper concept of point in relation to a line segment. The answers “more than 3” and “infinity” in Tests II and III are probably also related to the confusion of the terms circle and disc.

The students’ comments allow us to see their ways of thinking about a straight line. Comments 1B and 1D, which were most frequent in Task 1, refer to the potential infinity of a straight line and processing thinking (Monaghan, 2001). On the other hand, comments 1A, 2A and 2B could refer to actual infinity. The highest occurrence of such comments appeared in Test III. The prevalence of process thinking in the spirit of Euclid was also obvious from the pre-test interviews. In accordance with Boero, Douek and Garuti (2003), it is evident from the comments that some pupils finishing primary school already show very advanced thinking about infinity. Students who wrote both options (comment 2C) were obviously hesitant about trusting the picture or the word problem. Several students disagreed with the representation of a straight line in the picture (comments 2D, 2E). These students are convinced that a straight line must be drawn from one edge to the other edge of a given area. They likely learned this at school, as this is the method of representing a straight line recommended by textbooks. Another influence of school education was noticed in the pre-test interviews. Some students argued that a straight line can be distinguished from a line segment by their names (whether a lowercase or a capital letter is used). Thus, instead of focusing on the characteristics of straight lines or line segments, they focused on formal knowledge. The students who drew the picture like the one on the right in Figure 2 can imagine the existence of two infinities on a straight line; Jirotková and Litter (2004) also encountered this idea among students.

The concept of infinity is complicated and it is not possible to understand it within a short time (Jirotková & Litter, 2004). A highly abstract concept such as a straight line cannot be precisely defined in secondary education or even earlier, because it resulted in pupils´ formalism (Rendl et al., 2013). On the other hand, according to Fischbein, Tirosh and Hess (1979), the intuitive concept of infinity is developed by grade 7 at the latest, and further mathematics training affects only the formal understanding of the infinity concept. A straight line is a visual geometrical object suitable for illustrating infinity in mathematics. By means of a straight line we can define other geometrical concepts such as a half-line, a half-plane, an angle, etc. In our opinion, intuitive understanding of the infinity of a straight line in primary school and its strengthening in the succeeding levels of education is very important. A pupil should understand potential infinity in the spirit of Euclid by the end of lower secondary education. The role of upper secondary education is to create an idea of the actual infinity of a straight line.

The findings mentioned above imply following these recommendations: It is suitable to foster an intuitive concept of a straight line in primary school. Particular emphasis should be placed on the characteristics which distinguish it from a line segment; not on formal knowledge such as its name. In the succeeding levels of education, students should repeatedly work with the concept of the straight line infinity. Teachers must not depend on the students' full understanding of the concept of a straight line. The difference between a line segment and a straight line can be strengthened using tasks in which the pictured part of a straight line needs to be extended. In contemporary Czech textbooks, these tasks are virtually absent. We have found only one, which is similar to Task 1, in (Eichlerová, Staudková & Vlček, 2013).

The research confirmed our hypothesis, based on our pedagogical experience, that most students are not aware of the possibility of extending part of a given straight line. On the other hand, many of them visualized the extension of it and some students also gave interesting reasons for their solutions. Understanding the concept of a straight line is important, e.g., for understanding the number system or for preventing skipping a possible solution to a task on geometrical construction. We particularly consider a full understanding of this concept to be necessary for pre-service teachers who should motivate and inspire their future pupils.

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References

Boero, P., Douek, N. & Garuti, R. (2003). Children's concepts of infinity of numbers in a grade 5 classroom discussion context. In N. A. Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 121-128). Honolulu, USA: PME

Divíšek, J., Hošpesová, A. & Kuřina, F. (2000). *Svět čísel a tvarů. Metodická příručka k výuce matematiky ve 4. ročníku základní školy*. Praha, CZE: Prometheus.

Eichlerová, M., Staudková, H. & Vlček, O. (2013). *Matematika pro 2. (3.) ročník ZŠ, sešit 7.* Všeň, CZE: Alter.

Fischbein, E. (2001). Tacit models and infinity. *Educational Studies in Mathematics, 48(2–3),* 309-329. https://doi.org/10.1023/A:1016088708705

Fischbein, E., Tirosh, D., & Hess, P. (1979). The Intuition of Infinity. *Educational Studies in Mathematics, 10(1)*, 3-40. https://doi.org/10.1007/BF00311173

Hannula, M. S., Pehkonen, E., Maijala, H. & Soro, R. (2006). Levels of students' understanding on infinity. *Teaching Mathematics and Computer Science, 4(2)*, 317-337. https://doi.org/10.5485/TMCS.2006.0129

Jirotková, D., & Littler, G. (2003). Student's concept of infinity in the context of a simple geometrical construct. In N. A. Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 123-132). Honolulu, USA: PME.

Jirotková, D., & Littler, G. (2004). Insight into pupil's understanding of infinity in a geometrical context. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 97-104). Bergen, Norway: PME.

Monaghan, J. (2001). Young peoples' ideas of infinity. *Educational Studies in Mathematics 48(2-3)*, 239 -257. https://doi.org/10.1023/A:1016090925967

Moreno, L. E., & Waldegg, G. (1991). The conceptual evolution of actual mathematical infinity. *Educational Studies in Mathematics 22(3)*, 211-231. https://doi.org/10.1007/BF00368339

MŠMT. (2017). *Rámcový vzdělávací program pro základní vzdělávání*. Praha, CZE: VÚP.

Rendl, M., et al. (2013). *Kritická místa matematiky na základní škole očima učitelů.* Praha, CZE: Univerzita Karlova, Pedagogická fakulta.

Robová, J., Moravcová, V., Halas, Z., & Hromadová, J (2019). Žákovské koncepty trojúhelníku na začátku druhého stupně vzdělávání. *Scientia in educatione, 10(1)*, 1-22. https://doi.org/10.14712/18047106.1211

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